From PVMs to POVMs

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Abstract

We review projection-valued measures (PVMs) and positive-operator-valued measures (POVMs), clarify their relationship via Naimark's dilation theorem, and collect useful state-reconstruction formulas. For informationally complete POVMs we give the dual-frame linear inversion (basis free) and the closed form for SIC-POVMs. For PVM-based tomography we show how a tomographically complete set of projective bases (e.g. Pauli for qubits, MUBs in prime-power dimensions) yields simple reconstruction. We also include convex physical estimators (least-squares and maximum likelihood).

1 Preliminaries and notation

Let $\mathcal{H} \simeq \mathbb{C}^d$ be a d-dimensional Hilbert space. Density operators are positive semidefinite (PSD) $\rho \succeq 0$ with $\operatorname{Tr}\rho = 1$. We use the Hilbert–Schmidt inner product $\langle X, Y\rangle = \operatorname{Tr}(X^\dagger Y)$ on $\mathsf{Herm}(\mathcal{H})$.

2 Projective measurements (PVMs)

Definition 1 (PVM)**.** A *projection-valued measure (PVM)* $\{P_i\}_{i\in\mathcal{M}}$ on $\mathcal{H}$ satisfies $P_i = P_i^\dagger$, $P_i^2 = P_i$, $P_iP_j = \delta_{ij}P_i$, and $\sum_i P_i = \mathbb{1}$. Measuring ρ yields probabilities $p_i = \operatorname{Tr}(P_i\rho)$ and post-measurement state $\rho_i' = \frac{P_i\rho P_i}{\operatorname{Tr}(P_i\rho)}$.

For a single orthonormal basis $\{|\phi_i\rangle\}$, $P_i = |\phi_i\rangle\langle\phi_i|$. A single PVM has at most d outcomes and is generally *not* informationally complete (IC) for state tomography; IC requires enough different projective settings (see §6).

2.1 Concrete PVM Examples

In each example, the Born rule is $p_i = \operatorname{Tr}(P_i\rho)$ and the post-measurement state for outcome i is

$$\rho_i' = \frac{P_i\rho P_i}{\operatorname{Tr}(P_i\rho)}.$$

All families $\{P_i\}$ below satisfy $P_i = P_i^\dagger = P_i^2$, $P_iP_j = \delta_{ij}P_i$, and $\sum_i P_i = \mathbb{1}$.

2.2 Computational-basis PVM on a qudit

On $\mathcal{H} \cong \mathbb{C}^d$ with orthonormal basis $\{|j\rangle\}_{j=0}^{d-1}$,

$$P_j = |j\rangle\langle j|, \quad j = 0, \ldots, d-1, \qquad \sum_{j=0}^{d-1} P_j = \mathbb{1}.$$

For a pure state $|\psi\rangle = \sum_j \alpha_j |j\rangle$, $p_j = |\alpha_j|^2$ and $\rho_j' = |j\rangle\langle j|$.

2.2.1 Qubit Pauli PVMs

For $L \in \{X, Y, Z\}$ with Pauli matrices σ_L, define

$$P_\pm^{(L)} = \frac{\mathbb{1} \pm \sigma_L}{2}.$$

If $\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma})$ with Bloch vector $\vec{r} = (r_X, r_Y, r_Z)$, then

$$p_\pm^{(L)} = \mathrm{Tr}\Big(P_\pm^{(L)} \rho\Big) = \frac{1 \pm r_L}{2}, \qquad \rho'_\pm = |\pm_L\rangle\langle\pm_L| \,.$$

Explicitly,

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Are Pauli measurements a PVM? For a fixed Pauli observable $L \in \{X, Y, Z\}$, the two effects

$$P_\pm^{(L)} = \tfrac{1}{2}\big(I \pm \sigma_L\big)$$

form a PVM: $P_\pm^{(L)} = P_\pm^{(L)\dagger}$, $(P_\pm^{(L)})^2 = P_\pm^{(L)}$, $P_+^{(L)} P_-^{(L)} = 0$, and $P_+^{(L)} + P_-^{(L)} = I$. However, projectors from *different* Pauli settings are neither mutually orthogonal nor commuting in general. For example,

$$P_+^{(X)} P_+^{(Z)} = \tfrac{1}{4}(I + \sigma_X + \sigma_Z - i\,\sigma_Y) \neq 0,$$

so the union over $L \in \{X, Y, Z\}$ is *not* a single PVM.

It is common to aggregate the three settings into a single 6-outcome *POVM* by defining

$$E_\pm^{(L)} = \tfrac{1}{3}\, P_\pm^{(L)}, \qquad \sum_{L \in \{X,Y,Z\}} \sum_{\pm} E_\pm^{(L)} = I,$$

which is informationally complete for qubits. Tomographic inversion is then

$$r_L = 6\, p_+^{(L)} - 1, \qquad \hat{\rho} = \tfrac{1}{2}\left(I + \sum_{L \in \{X,Y,Z\}} r_L\, \sigma_L \right),$$

where $p_+^{(L)} = \mathrm{Tr}\Big(E_+^{(L)} \rho\Big)$.

2.2.2 Degenerate PVM: two-qubit parity

On $\mathcal{H} = (\mathbb{C}^2)^{\otimes 2}$, the parity observable $Z \otimes Z$ has eigenvalues ± 1 with degeneracy 2. The PVM is

$$P_{\text{even}} = |00\rangle\langle 00| + |11\rangle\langle 11| \,, \qquad P_{\text{odd}} = |01\rangle\langle 01| + |10\rangle\langle 10| \,.$$

This projects onto two-dimensional subspaces. The Lüders update preserves coherences *within* the parity subspace: $\rho \mapsto P_{\text{even}} \rho P_{\text{even}} / p_{\text{even}}$ or analogously for odd.

2.2.3 Total spin (singlet–triplet) PVM for two qubits

Equivalently, measure total angular momentum J^2:

$$\begin{aligned} \big|\psi^-\big\rangle &= \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \quad P_{\text{sing}} = \big|\psi^-\big\rangle\big\langle\psi^-\big| \,, \\ \big|\phi^\pm\big\rangle &= \tfrac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \ \big|\psi^+\big\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\ P_{\text{trip}} &= \big|\phi^+\big\rangle\big\langle\phi^+\big| + \big|\phi^-\big\rangle\big\langle\phi^-\big| + \big|\psi^+\big\rangle\big\langle\psi^+\big| \,. \end{aligned}$$

This is a two-outcome PVM with ranks 1 and 3 (degenerate triplet).

2.2.4 Joint PVM for commuting observables

If A and B commute and have spectral decompositions $A = \sum_a a\,\Pi_a$, $B = \sum_b b\,\Xi_b$, then

$$\{\,\Pi_a \Xi_b\,\}_{a,b}$$

is a PVM (orthogonal projectors onto joint eigenspaces). Example on two qubits: measuring $Z \otimes \mathbb{1}$ and $\mathbb{1} \otimes Z$ jointly yields $\{|00\rangle\langle 00|, |01\rangle\langle 01|, |10\rangle\langle 10|, |11\rangle\langle 11|\}$.

2.2.5 Fourier/DFT-basis PVM on a qudit

Let $\omega = e^{2\pi i/d}$ and define $|f_k\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \omega^{jk} |j\rangle$, $k = 0, \ldots, d-1$. Then $Q_k = |f_k\rangle\langle f_k|$ form a PVM. For prime d, the computational-basis PVM $\{P_j\}$ and the Fourier-basis PVM $\{Q_k\}$ are mutually unbiased: $|\langle j|f_k\rangle|^2 = 1/d$.

2.2.6 Bell-basis PVM (entangled basis)

On two qubits, the four Bell states $\{|\phi^\pm\rangle, |\psi^\pm\rangle\}$ give a rank-1 PVM

$$\{\Pi_{\phi^+}, \Pi_{\phi^-}, \Pi_{\psi^+}, \Pi_{\psi^-}\}, \qquad \Pi_\beta = |\beta\rangle\langle\beta|.$$

This PVM is central in Bell tests and teleportation.

2.2.7 Continuous-spectrum example: position PVM

On $\mathcal{H} = L^2(\mathbb{R})$, the position operator X has spectral measure $\{P_\Omega\}_{\Omega \subset \mathbb{R}}$ with

$$P_\Omega = \int_\Omega |x\rangle\langle x|\; dx, \qquad \int_{\mathbb{R}} |x\rangle\langle x|\; dx = \mathbb{1}.$$

For a wavefunction $\psi(x)$, $p(\Omega) = \mathrm{Tr}(P_\Omega\,\rho) = \int_\Omega |\psi(x)|^2\, dx$. Updates follow the same Lüders rule with projection onto $L^2(\Omega)$.

3 Generalized measurements (POVMs)

Definition 2 (POVM)**.** A *POVM* on a d-dimensional Hilbert space $\mathcal{H}$ is a finite (or countable) family of positive semidefinite operators $\{E_j\}_{j\in\mathcal{M}}$ such that $\sum_j E_j = \mathbb{1}$ (or $\mathbb{I}$). Given a state ρ, the probability of outcome j is

$$p(j \mid \rho) = \mathrm{Tr}(E_j \rho). \tag{1}$$

POVMs strictly generalize PVMs and can model coarse-graining, noise, or collective projective measurements on a larger space.

Post-measurement states and instruments. A POVM does not, by itself, fix the *post-measurement* state; one must specify a *quantum instrument* $\{\mathcal{M}_j\}_j$, i.e. completely positive (CP) maps with $\sum_j \mathrm{Tr}[\mathcal{M}_j(\rho)] = 1$ for all states. Every instrument admits a Kraus representation

$$\mathcal{M}_j(\rho) = \sum_\ell K_{j,\ell}\,\rho\,K_{j,\ell}^\dagger, \qquad E_j = \sum_\ell K_{j,\ell}^\dagger K_{j,\ell}. \tag{2}$$

Upon observing outcome j, the normalized posterior state is

$$\rho_j' = \frac{\mathcal{M}_j(\rho)}{\mathrm{Tr}(E_j \rho)}. \tag{3}$$

Different choices of $\{K_{j,\ell}\}$ (same E_j) generally produce different post-measurement states.

Square-root (Lüders-type) instrument. A canonical and often "least disturbing" choice is the single-Kraus update

$$\mathcal{M}_j(\rho) = E_j^{1/2}\,\rho\,E_j^{1/2}, \qquad \rho_j' = \frac{E_j^{1/2}\rho E_j^{1/2}}{\mathrm{Tr}(E_j\rho)}. \tag{4}$$

For PVMs (E_j are orthogonal projectors), (4) reduces to the usual Lüders projection $P_j\rho P_j/p(j)$.

Naimark view. Any POVM can be realized as a projective measurement on a larger space: there exist an ancilla $\mathcal{K}$, a unit ancilla state $|0\rangle$, a unitary U on $\mathcal{H}\otimes\mathcal{K}$, and a PVM $\{\Pi_j\}$ on $\mathcal{H}\otimes\mathcal{K}$ with

$$E_j = \mathrm{Tr}_{\mathcal{K}}\big[(\mathbb{1}\otimes|0\rangle\langle 0|)\,U^\dagger\Pi_j U\,(\mathbb{1}\otimes|0\rangle\langle 0|)\big].$$

The corresponding posterior is the reduced state obtained from $U(\rho\otimes|0\rangle\langle 0|)U^\dagger$ conditioned on Π_j.

Remark 1. Thus, any POVM can be realized as a projective measurement on a larger system, after coupling by U and discarding the ancilla. Rank-one POVMs arise from PVMs on minimal dilations.

3.1 Examples of POVMs

(a) Unsharp two-outcome qubit measurement

Fix a Bloch direction $\hat{n}$ and a sharpness parameter $0 \le \eta \le 1$.

$$E_\pm = \tfrac{1}{2}\big(\mathbb{1} \pm \eta\,\hat{n}\cdot\vec{\sigma}\big), \qquad E_+ + E_- = \mathbb{1},\ E_\pm \succeq 0.$$

For $\eta = 1$ this is the projective measurement along $\hat{n}$; for $\eta < 1$ it models classical/quantum noise. Square-root update uses (4) and preserves the symmetry about $\hat{n}$.

(b) Trine POVM on a qubit (equatorial, 3 outcomes)

Let $\{|t_k\rangle\}_{k=0}^{2}$ be three pure states with Bloch vectors 120° apart in the xy-plane. Define $E_k = \frac{2}{3}\,|t_k\rangle\langle t_k|$. Then $\sum_k E_k = \mathbb{1}$ and the POVM has three outcomes. It is informationally complete for states restricted to the equatorial plane, and is widely used in qubit discrimination on a great circle.

(c) Tetrahedral SIC-POVM (qubit, 4 outcomes)

Let $\{|\psi_j\rangle\}_{j=1}^{4}$ have Bloch vectors at the vertices of a regular tetrahedron. Set $E_j = \frac{1}{2}\,|\psi_j\rangle\langle\psi_j|$. The set is symmetric and *informationally complete*; given $p_j = \mathrm{Tr}(E_j\rho)$,

$$\rho = \sum_{j=1}^{4}\big(3p_j - \tfrac{1}{2}\big)\,|\psi_j\rangle\langle\psi_j|\,.$$

(See §4.1 for physical estimators with finite data.)

(d) Unambiguous discrimination of two nonorthogonal pure states

Let $|\psi\rangle$ and $|\phi\rangle$ be linearly independent qubit (or qudit-plane) states with overlap $c = |\langle\psi|\phi\rangle| \in (0,1)$. Choose unit vectors $|\psi_\perp\rangle$, $|\phi_\perp\rangle$ orthogonal to $|\psi\rangle$ and $|\phi\rangle$ within span$\{\psi,\phi\}$. For any $0 < \alpha \le \frac{1}{1+c}$ the three effects

$$E_\psi = \alpha\;|\phi_\perp\rangle\langle\phi_\perp|\,, \qquad E_\phi = \alpha\;|\psi_\perp\rangle\langle\psi_\perp|\,, \qquad E_? = \mathbb{1} - E_\psi - E_\phi \succeq 0$$

form a POVM with the property: $\mathrm{Tr}(E_\psi\,|\phi\rangle\langle\phi|) = 0$ and $\mathrm{Tr}(E_\phi\,|\psi\rangle\langle\psi|) = 0$, so conclusive outcomes never misidentify the state; $E_?$ is "inconclusive." For equal priors, the optimal choice of α yields success probability $1 - c$.

(e) Coarse-graining / classical noise on a PVM

Starting from a two-outcome PVM $\{P_0, P_1\}$, a binary symmetric classical noise channel with flip probability ε produces a POVM

$$F_0 = (1-\varepsilon)P_0 + \varepsilon P_1, \qquad F_1 = \varepsilon P_0 + (1-\varepsilon)P_1,$$

with $F_0 + F_1 = \mathbb{1}$ but F_i not idempotent unless $\varepsilon \in \{0, 1\}$.

(f) Continuous-variable example: heterodyne (coherent-state) POVM

On a single bosonic mode, the coherent states $\{|\alpha\rangle\}_{\alpha\in\mathbb{C}}$ resolve the identity:

$$\frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha| \, d^2\alpha = \mathbb{1}.$$

The heterodyne POVM has density $E(\alpha) = \frac{1}{\pi}|\alpha\rangle\langle\alpha|$; the outcome distribution is the Husimi–Q function $Q_\rho(\alpha) = \mathrm{Tr}(E(\alpha)\rho)$.

Note: In this part, we will talk about later in the Haar measurement.

Remark (IC POVMs for tomography). Examples (c) and the 6-outcome "Pauli-6" POVM $E_\pm^{(L)} = \frac{1}{3}(\mathbb{1} \pm \sigma_L)/2$ (for $L = X, Y, Z$) are informationally complete on a qubit. Use the linear/dual-frame or convex estimators from §4.1 to reconstruct ρ from empirical frequencies.

4 Informational completeness and linear inversion

A POVM $\{E_j\}_{j=1}^m$ is *informationally complete (IC)* if the real span of $\{E_j\}$ is $\mathsf{Herm}(\mathcal{H})$ (dimension d^2). Then the linear map $X \mapsto (\mathrm{Tr}(E_j X))_{j=1}^m$ is injective, and ρ is uniquely determined by $p_j = \mathrm{Tr}(E_j\rho)$.

Define the *frame operator* $\mathcal{S} : \mathsf{Herm}(\mathcal{H}) \to \mathsf{Herm}(\mathcal{H})$,

$$\mathcal{S}(X) = \sum_{j=1}^m \mathrm{Tr}(E_j X)\, E_j.$$

If $\{E_j\}$ is IC, $\mathcal{S}$ is invertible. The *canonical dual frame* is $D_j = \mathcal{S}^{-1}(E_j)$ and yields the basis-free linear inversion

$$\rho = \sum_{j=1}^m p_j\, D_j, \qquad p_j = \mathrm{Tr}(E_j\rho). \tag{5}$$

Equivalently, choose any Hilbert–Schmidt orthonormal operator basis $\{B_\alpha\}_{\alpha=0}^{d^2-1}$ with $B_0 = \mathbb{1}/\sqrt{d}$, expand $\rho = \sum_\alpha r_\alpha B_\alpha$, set $A_{j\alpha} = \mathrm{Tr}(E_j B_\alpha)$, and solve $r = A^+ p$ (Moore–Penrose pseudoinverse).

4.1 Finite data and physical estimators

With N samples, frequencies $f_j = n_j/N$ plugged into (5) give an unbiased but possibly non-PSD $\hat\rho_{\mathrm{lin}}$. Two standard convex fixes:

$$\textbf{(Constrained LS)} \quad \min_{\rho\succeq 0,\ \mathrm{Tr}\,\rho=1} \sum_j \big(\mathrm{Tr}(E_j\rho) - f_j\big)^2,$$

$$\textbf{(MLE)} \quad \max_{\rho\succeq 0,\ \mathrm{Tr}\,\rho=1} \sum_j n_j \log \mathrm{Tr}(E_j\rho).$$

5 Special IC measurements and closed forms

SIC-POVM. A symmetric informationally complete POVM consists of $m = d^2$ rank-1 effects $E_j = \frac{1}{d}\Pi_j$ with $\Pi_j = |\psi_j\rangle\langle\psi_j|$ and $\mathrm{Tr}(\Pi_j\Pi_k) = \frac{d\delta_{jk}+1}{d+1}$. Then the inversion is *closed form*:

$$\rho = \sum_{j=1}^{d^2} \left((d+1)p_j - \frac{1}{d} \right) \Pi_j. \tag{6}$$

Pauli-6 / Tetrahedron (qubit). For $d = 2$, the tetrahedral SIC has 4 outcomes $\{\Pi_j\}_{j=1}^4$ and (6) becomes $\rho = \sum_j (3p_j - \frac{1}{2})\Pi_j$.

6 Reconstruction from PVMs

A single PVM is not IC for $d > 1$. However, a *set* of PVMs can be IC:

- **Qubits.** Measure the three Pauli PVMs Z, X, Y on three equally sized blocks of copies. If $p_L^{(+)}$ is the probability of the +1 outcome for $L \in \{X, Y, Z\}$, the Bloch vector is $r_L = 2p_L^{(+)} - 1$, and
$$\hat{\rho} = \tfrac{1}{2}\big(\mathbb{1} + r_X\sigma_X + r_Y\sigma_Y + r_Z\sigma_Z\big).$$
- **MUBs (prime-power d).** A complete set of $d+1$ mutually unbiased bases gives $d(d+1)$ outcomes arranged in $d+1$ PVMs and is IC. Linear inversion amounts to projecting onto basis projectors and solving a full-rank linear system; MLE/LS as in §4.1.

7 Worked example (qubit, arbitrary IC POVM)

Let $\{E_j\}_{j=1}^m$ be IC on a qubit. Choose the Pauli basis $B_0 = \mathbb{1}/\sqrt{2}$, $B_k = \sigma_k/\sqrt{2}$. Form $A_{j\alpha} = \mathrm{Tr}(E_j B_\alpha)$ and solve $r = A^+ f$ from frequencies f. Then

$$\hat{\rho}_{\mathrm{lin}} = \tfrac{1}{2}\Big(\mathbb{1} + \sum_{k=1}^{3} r_k\sigma_k\Big),$$

followed by the PSD projection (LS or MLE) if needed.

8 Discussion

POVMs provide hardware-friendly, noise-robust measurements and allow elegant linear inversions via dual frames; PVMs remain sufficient when combined into IC sets (Pauli, MUBs). In practice, prefer convex physical estimators and, when possible, exploit symmetry (e.g. SIC) for variance reduction.

References